Weight Optimization of the Postbuckled Integrally Stiffened Wide Column

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An optimum stress equation for an integrally stiffened wide column is developed which parallels the buckle resistant simultaneous mode analysis but which also holds for postbuckling design. Coupling between the local failure and Euler buckling stress is accounted for by a parabolic fit interaction equation where failure stress is computed using an empirically modified effective width theory. It is shown that the ratio of applied stress to local buckling stress can be considered an optimized parameter. Specific numerical results for aluminum and titanium demonstrate that at sufficiently low values of the load index, the postbuckled design can represent a weight savings. It is shown that these index values correspond to applied stresses approximately one half the respective proportional limits. The analysis is limited to optimum values of the ratio of applied stress to local buckling stress which do not exceed 1.55 due to limitations on the interaction equation. Therefore, the postbuckling weight advantage cannot be fully explored at the lower index range. The determination of the upper bound load index where postbuckling design ceases to represent a weight advantage is well within the applicability of the interaction equation and is the principal conclusion of this investigation.

Nomenclature

b_s = distance between stiffeners
b_w = stiffener web height
C_{ei} = failure stress equation coefficient for the ith element of the cross section
C_{eT} = equivalent coefficient for the total cross section
E = modulus of elasticity
K = coupled local buckling coefficient
L = column length
N_x = distributed load (lb/in.)
P_T = total failure load of representative width
r_b = b_w/b_s
r_t = t_w/t_s
t_s = skin thickness
t_w = web thickness
$\bar{t}$ = weight effective thickness (cross-sectional area per unit width)
ϵ = efficiency factor in the efficiency equation
η_T = plasticity correction (E_T/E)
ψ_{cc} = slack variable equal to the ratio σ_A/σ_{cc}
ψ_E = slack variable equal to the ratio σ_A/σ_E
ψ_L = slack variable equal to the ratio σ_A/σ_L
μ = poisson's ratio (assumed equal to 0.3)
ρ = radius of gyration
σ_A = average applied stress = $N_x/\bar{t}$
σ_E = Euler stress
σ_L = local buckling stress
σ_{cc} = average failure stress of the total cross section
σ_{cci} = average failure stress of the ith element of the cross section
σ_{cy} = compressive yield stress
Φ = a function of material parameters and r_b and r_t with the dimensions of stress

Introduction

CONSIDERABLE controversy has been generated over the past few decades concerning the possibility of postbuckling design criteria as being competitive to the simultaneous buckling mode approach for minimum weight of compression structures. Shanley,[1] one of the early pioneers of simultaneous mode design, has been the most outspoken proponent for the use of buckle resistant criteria for the prediction of local failure. He showed that a square tube as a Euler strut would be heavier when designed on a postbuckling basis as opposed to buckle resistant, for all realistic load index levels.[2] Many, however, have been reluctant to extrapolate this result to some of the more complex compression structures, such as stiffened skin cross sections.

In what follows, an indiced minimum weight analysis will be developed which parallels the simultaneous mode approach but which admits to the possibility of applied stress exceeding the local buckling stress as an optimized parameter dependent on the load index. The analysis is applied principally to the integrally stiffened wide column but can be readily adapted to most wide column concepts.

Development of a Generalized Optimum Stress Equation

The Euler-Engesser stress for wide columns under axial compression is given by the well-known expression

$$\sigma_E = \pi^2 \eta_T E / [(1 - \mu^2)(L/\rho)^2] \tag{1}$$

For the integrally stiffened cross section shown in Fig. 1, the radius of gyration ρ can be expressed in terms of the stiffener separation b_s, the ratios of web to skin thickness $r_t = t_w/t_s$ and lengths $r_b = b_w/b_s$ as follows

$$\rho^2 = b_s^2 f(r_t, r_b) \tag{2}$$

where

$$f(r_t,r_b) = r_t r_b^3/[3(1 + r_t r_b)] - r_t^2 r_b^4/[4(1 + r_t r_b)^2]$$

Combining Eqs. (1) and (2) yields the Euler stress in terms of the cross-section variables

$$\sigma_E = \pi^2 \eta_T E b_s^2 f(r_t, r_b)/[(1 - \mu^2)L^2] \tag{3}$$

For coupled local buckling of an integrally stiffened cross section, the following equation can be written in terms of the

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thickness to length ratio of the skin

$$\sigma_L = K\pi^2\eta_T{}^{1/2}E/[12(1-\mu^2)]\,(t_s/b_s)^2 \tag{4}$$

where K is a function of r_b and r_t.[3]

The applied stress can be written in terms of the applied line load distribution N_x and cross-section variables as follows

$$\sigma_A = N_x/\bar{t} = N_x/[t_s(1 + r_t r_b)] \tag{5}$$

At this point, a buckle resistant approach would proceed by equating the Euler, local and applied, stresses. The applied stress (which is to be maximized) could then be expressed in terms of a load index (N_x/L) and a function of r_b and r_t. The optimization would be completed by determining the optimum values of r_b and r_t such that the applied stress would be maximum for a given load index.

A similar index formulation can be achieved without the buckle resistant restriction by introducing "slack variables," ψ_L and ψ_E such that

$$\sigma_A = \psi_L\sigma_L = \psi_E\sigma_E \tag{6}$$

Employing Eq. (6), the Euler, local and applied stress, [Eqs. (3–5)] can be combined, and taking $\mu = 0.3$, yields

$$\sigma_A = \{3.13\ K^{1/2}[f(r_t,r_b)]^{1/2}/(1 + r_t r_b)\}^{1/2} \times (\psi_L\psi_E)^{1/4}\eta_T{}^{3/8}E^{1/2}\ (N_x/L)^{1/2} \tag{7}$$

A result identical to a buckle resistant analysis, with the exception of the factors $(\psi_L\psi_E)^{1/4}$.

In previous papers[4] the bracketed quantity (a function of r_t and r_b only) has been denoted by the symbol ϵ and termed the "efficiency factor." For the sake of convenience, we make the same definition recognizing, however, that the total measure of efficiency is now $\epsilon^{1/2}(\psi_L\psi_E)^{1/4}$. Rewriting Eq. (7) in this form yields

$$\sigma_A = \epsilon^{1/2}(\psi_L\psi_E)^{1/4}\eta_T{}^{3/8}E^{1/2}(N_x/L)^{1/2} \tag{8}$$

It can be seen that the most efficient design results when the quantity $\epsilon^{1/2}(\psi_L\psi_E)^{1/4}$ is a maximum. It should be emphasized at this time that Eq. (8) is completely general. No assertions as to how ψ_L and ψ_E might be specified have been made. That is to say, no statement concerning the interaction between σ_A, σ_L, and σ_E has been made. In order to formulate such a statement, consideration must be given to the possible dependence of ψ_E on a selection of ψ_L which exceeds unity. This dependence results from the coupling between modes that exists when the applied stress exceeds the local buckling stress.

Local and General Mode Coupling

An interaction equation of a parabolic fit type has been recommended by Gerard and is well correlated for stiffened skin wide column configurations.[5] The equation requires that the column buckling allowable be bounded from above by the average failure stress of the cross section based on effective width theory and can only approach this value as $L/\rho \to 0$ (see Fig. 2). This equation can be expressed as follows:

$$\sigma_A/\sigma_{cc} = 1 - [1 - (\sigma_L/\sigma_{cc})]\ \sigma_L/\sigma_E \tag{9}$$

Besides σ_A, σ_E, and σ_L, Eq. (9) contains the average failure stress of the cross-section σ_{cc}. Again, avoiding any re-

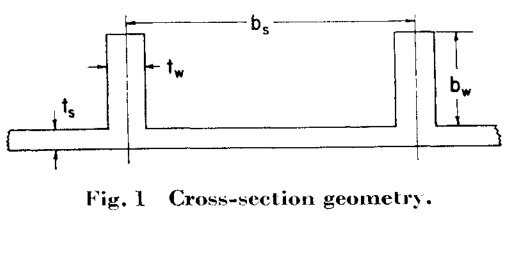

Fig. 1 Cross-section geometry.

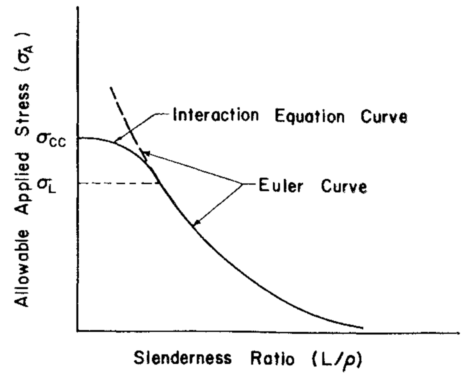

Fig. 2 Interaction equation.

strictions on the value of applied stress a slack variable is introduced such that

$$\sigma_A = \psi_{cc}\sigma_{cc} \tag{10}$$

Now, inserting the slack variables [Eqs. (6) and (10)] into the interaction equation [Eq. (9)] yields

$$\psi_E = \psi_L{}^2[(1 - \psi_{cc})/(\psi_L - \psi_{cc})] \tag{11}$$

Equation (11) demonstrates the dependence of ψ_E on ψ_L and ψ_{cc}. By consideration of the failure stress equation, this dependence will be reduced to ψ_L alone.

Cross-Section Failure Stress

Consider a representative element of skin and stiffener as shown in Fig. 1. According to Needham's empirical modification of effective width theory, the average failure stress associated with the ith element can be expressed as[6]

$$\sigma_{cc_i} = C_{e_i}(\sigma_{cy}E)^{1/2}(t_i/b_i)^{3/4} \tag{12}$$

where C_{e_i} is a constant which depends on condition of support and material. The load carried by the integral web is

$$P_w = \sigma_{cc_w}t_w b_w \tag{13}$$

and for the skin

$$P_s = \sigma_{cc_s}t_s b_s \tag{14}$$

The total load on the representative element is the sum of Eqs. (13) and (14)

$$P_T = \sigma_{cc_s}t_s b_s + \sigma_{cc_w}t_w b_w \tag{15}$$

Defining the average failure stress σ_{cc} associated with the total cross section as

$$\sigma_{cc} = \frac{\text{Failure Load on Representative Element}}{\text{Area of Representative Element}} = \frac{P_T}{b_s\bar{t}} \tag{16}$$

it follows from Eqs. (15) and (16) that

$$\sigma_{cc} = (\sigma_{cc_s}t_s b_s + \sigma_{cc_w}t_w b_w)/b_s\bar{t} \tag{17}$$

The effective thickness $\bar{t}$ can be written as $t_s(1 + r_t r_b)$. Therefore, Eq. (17) becomes

$$\sigma_{cc} = (\sigma_{cc_s} + r_t r_b\sigma_{cc_w})/(1 + r_t r_b) \tag{18}$$

From Eq. (12),

$$\begin{aligned}\sigma_{cc_s} &= C_{e_s}(\sigma_{cy}E)^{1/2}(t_s/b_s)^{3/4}\\ \sigma_{cc_w} &= C_{e_w}(\sigma_{cy}E)^{1/2}(t_w/b_w)^{3/4}\end{aligned} \tag{19}$$

Now, (t_w/b_w) can be written as

$$t_w/b_w = (r_t/r_b)\ (t_s/b_s) \tag{20}$$

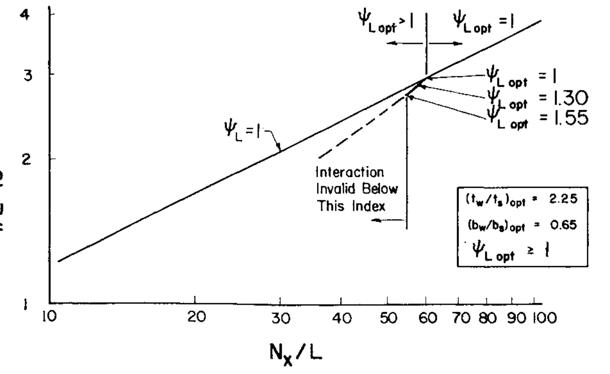

Fig. 3 Buckle resistant vs postbuckling design; integrally stiffened wide column; aluminum 7075-T6.

Therefore, combining Eqs. (18–20) yields

$$\sigma_{cc} = C_{eT}(\sigma_{cy}E)^{1/2}(t_s/b_s)^{3/4} \tag{21}$$

where

$$C_{eT} = (C_{e_s} + r_t^{7/4}r_b^{1/4}C_{ew})/(1 + r_tr_b)$$

as the failure stress equation for the combination of skin and web stiffener. C_{eT} is the failure constant for the combination in terms of the individual constants and the length and thickness ratios.

Equation (21) can be reduced to an expression relating ψ_L and ψ_{cc} by first combining Eqs. (4) and (6) to yield for the t_s/b_s ratio

$$t_s/b_s = 1.05(\sigma_A/K\eta_T^{1/2}E\psi_L)^{1/2} \tag{22}$$

Now combining Eqs. (21) and (22), with σ_{cc} replaced by σ_A/ψ_{cc} [Eq. (10)] yields

$$\psi_{cc} = (\eta_T^{3/16}\sigma_A^{5/8}/\phi^{5/8})\ \psi_L^{3/8} \tag{23}$$

where

$$\phi = 1.06\ C_{eT}^{8/5}\sigma_{cy}^{4/5}E^{1/5}/K^{3/5}$$

and where ϕ depends only on material, r_b and r_t. Equation (23) demonstrates that ψ_{cc} can be written as a material and stress dependent function of ψ_L.

Optimization Procedure

Equation (11) and (23) show that, if account is taken of general mode coupling and cross-section failure, ψ_E cannot be chosen independently of ψ_L. The result is that the efficiency measure of Eq. (8) can be expressed in terms of three independent parameters, vis., r_t, r_b, and ψ_L, subject to optimization. Combining Eqs. (8, 11, and 23) results in

$$\sigma_A = \epsilon^{1/2}\psi_L^{3/4}\left[\frac{1-(\eta_T^{3/16}\sigma_A^{5/8}\psi_L^{3/8}/\phi^{5/8})}{\psi_L-(\eta_T^{3/16}\sigma_A^{5/8}\psi_L^{3/8}/\phi^{5/8})}\right]\eta_T^{3/8}E^{1/2}\left(\frac{N_x}{L}\right)^{1/2} \tag{24}$$

Since the optimum stress equation is implicit in σ_A the optimization must be effected by selecting values for σ_A and then determining the optimum values of r_t, r_b, and ψ_L such that the corresponding load index would be a minimum. In this way, for an arbitrary selection of the load index, the optimum applied stress would be a maximum.

Results and Limitations

Equation (24) was programed according to the previous procedure for a representative alloy of both aluminum and titanium. The curves thus generated are shown in Figs. 3 and 4 in terms of an effective thickness index. It can be seen that the postbuckled design represents a weight advantage over the buckle resistant design for sufficiently low values of the load index, the cutoffs being 59 psi and 220 psi for the aluminum and titanium, respectively. These indices correspond to applied stresses of 20,100 psi for the aluminum and 49,500 psi for the titanium, approximately one half of the respective proportional limits.

As the load index is varied to values less than the previous cutoffs, the weight advantage can be seen to grow as does the optimum values of ψ_L. Very soon, however, a load index is reached where the interaction equation ceases to be valid. At sufficiently high optimum values of ψ_L, the interaction equation literally "overlaps" the Euler curve. This limitation on the applicability of the interaction equation is illustrated in Fig. 5 and is considered in detail in the Appendix. The postbuckling weight advantage is on the order of 2% at this point ($\psi_{L\text{opt}} = 1.55$, $\psi_E = 0.72$). From this point and below, the postbuckling curve is dashed to indicate an extrapolation of the trend exhibited in the region where the interaction equation was valid. It is interesting to note that the values of r_b and r_t which are optimum in the postbuckling region are those which are optimum for buckle resistant design.

Despite the limitation of the analysis to a restricted range for ψ_L (for both materials ψ_L had to be less than 1.55 to have a valid interaction equation), it is shown, if one accepts the interaction and failure criteria, that definite load index bounds exist where the postbuckling design does not represent a weight advantage over the buckle resistant design. That these bounds are considerably below the proportional limit for both materials is considered by the author to be the principal result of the analysis. The accuracy of the specific numerical results must hinge on empirical verification since the two empirical equations (interaction and failure stress evaluations) will introduce an approximation into the analysis commensurate with the scatter that usually accompanies such expressions. However, independent of failure and interaction criteria, the dependence of optimum applied stress on the product $(\psi_L\psi_E)^{1/4}$ [Eq. (8)] demonstrates that an increase in applied stress over the local buckling stress does not result in a proportionate decrease in weight. For example, if $\sigma_{A\text{opt}} = 2\sigma_L$, ($\psi_L = 2$), the optimum applied stress increases at most by a factor of $(2)^{1/4} = 1.19$. Recognize that this figure must be further reduced, since ψ_E would be less than unity as a result of general mode coupling. These results apply to compression structure in general.[7]

Appendix: Interaction Equation Limitation

The question is now raised as to whether an upper bound exists for ψ_L for a valid interaction equation and, if so, how does this bound vary with the load index or applied stress. Consider Fig. 5, which shows two possible relations the parabolic fit interaction equation can have with respect to the Euler column curve. Obviously, condition B, although mathematically consistant with Eq. (9), is a misrepresenta-

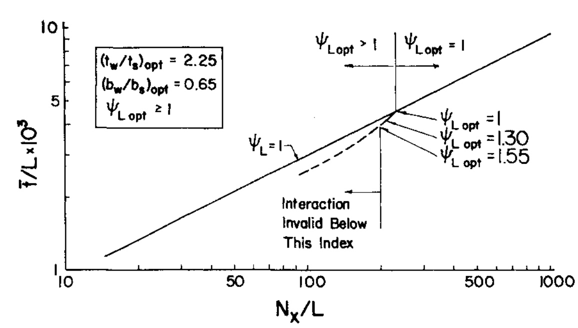

Fig. 4 Buckle resistant vs postbuckling design; integrally stiffened wide column; Ti-6Al-4V H.T.

tion of reality and we can only expect a valid interaction if the situation is as depicted in condition A. In order to insure this, we must establish σ_I as being either coincident with or falling below σ_L; viz.

$$\sigma_I \leq \sigma_L \quad \text{(A1)}$$

where σ_I is the solution to the equation which results when σ_I is equated to σ_E with the additional condition that σ_I satisfies the interaction equation as follows:

$$\sigma_I/\sigma_{cc} = [1 - (\sigma_L/\sigma_{cc})]\, \sigma_L/\sigma_E \quad \text{(A2)}$$

and since for an intersection with the column curve it is required that $\sigma_I = \sigma_E$, Eq. (A2) becomes

$$\sigma_I{}^2 - \sigma_{cc}\sigma_I + (\sigma_{cc}\sigma_L - \sigma_L{}^2) = 0 \quad \text{(A3)}$$

which factors to

$$(\sigma_I - \sigma_L)\,(\sigma_I + \sigma_L - \sigma_{cc}) = 0$$

and yields the two roots $\sigma_{I_1} = \sigma_L$ and $\sigma_{I_2} = (\sigma_{cc} - \sigma_L)$. Hence, Eq. (A1) requires that

$$\sigma_{cc} - \sigma_L \leq \sigma_L \quad \text{(A4)}$$

or

$$\sigma_L \geq \tfrac{1}{2}\sigma_{cc}$$

which states that the initial buckling stress must be at least as large as one half of the failure stress of the cross section. An attempt to specify σ_A and σ_L such that Eq. (A4) is violated will result in a design for which the assumed interaction equation is no longer applicable. Equation (A4) can be used to solve specifically for a bounded value of ψ_L in terms of stress. Using the definitions of the slack variables, Eq. (A4) becomes

$$\psi_L \leq 2\psi_{cc} \quad \text{(A5)}$$

and since the extreme value of ψ_{cc} is unity, it can be seen immediately that, for any stress value, ψ_L cannot exceed 2. In particular, since ψ_{cc} depends on the applied stress and and ψ_L, by virtue of Eq. (22), it is found that

$$\psi_L \leq 3.03\ \sigma_A/\phi \quad \text{(A6)}$$

Consequently an optimum value of ψ_L which violates the constraint of Eq. (A6) will result in an invalid interaction equation. Figures 3 and 4 indicate that this constraint limited the optimum value of ψ_L to under 1.55.

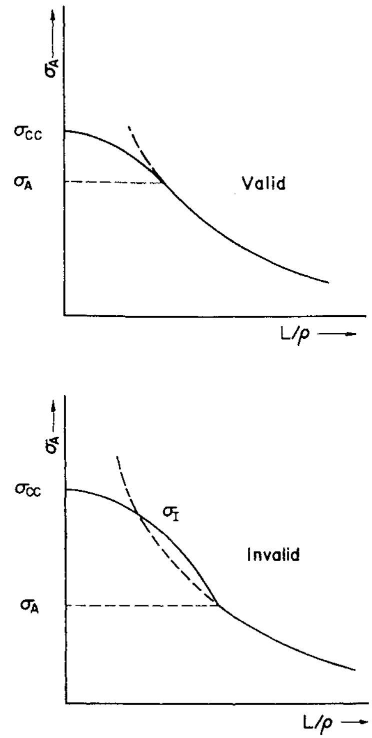

Fig. 5 Interaction equation limitation.

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